

Preferences in Quantum Games

Simon J.D. Phoenix & Faisal Shah Khan

Khalifa University, PO Box 127788, Abu Dhabi, UAE

February 24, 2015

Abstract

A quantum game can be viewed as a state preparation in which the final output state results from the competing preferences of the players over the set of possible output states that can be produced. It is therefore possible to view state preparation in general as being the output of some appropriately chosen (notional) quantum game. This reverse engineering approach in which we seek to construct a suitable notional game that produces some desired output state as its equilibrium state may lead to different methodologies and insights. With this goal in mind we examine the notion of preference in quantum games since if we are interested in the production of a particular equilibrium output state, it is the competing preferences of the players that determine this equilibrium state. We show that preferences on output states can be viewed in certain cases as being induced by measurement with an appropriate set of numerical weightings, or payoffs, attached to the results of that measurement. In particular we show that a distance-based preference measure on the output states is equivalent to a having a strictly-competitive set of payoffs on the results of some measurement.

1 Introduction

Whilst there are important issues to resolve in our full understanding of the relationship between correlation, entanglement and non-classical properties, it is clear that quantum mechanics admits possibilities for behaviours that cannot be achieved by any purely classical system. The introduction of quantum mechanical ideas in game theory is one example where entanglement offers the potential to achieve game results that cannot be obtained when playing games with classical objects obeying the laws of classical physics [1,2].

In our previous work we have argued for the importance of the notion of *playable* games; that is, games that can actually be played, or implemented, with objects that have a physical reality and obey physical laws [3-5]. This perspective allows a general framework for the description of a game (either quantum or classical) to be developed in which the comparisons between classical and quantum behaviours can more easily be drawn. In this perspective a game,

either quantum or classical, can be seen as a *state preparation* where the state that is actually prepared is a function of the initial state, the actions available to the players and their competing preferences over the possible states that can be produced.

If games are *state preparations*, then might not the converse be true? Can we view state preparations as the output of certain quantum games? If this is possible then we can view state preparations from a game-theoretic perspective. Instead of asking the question whether quantum games give us results unobtainable in classical games, the shift in view to state preparation asks the question whether we can re-interpret quantum behaviours in a game-theoretic way. In our previous work we have expressed this notion as that of ‘gaming the quantum’ rather than the usual approach in quantum games of ‘quantizing the game’ [3]. In this way we ask whether the tools of game theory can be applied to quantum mechanics, just as they are applied to fields like economics or biology, in order to give us different insights or different computational methodologies.

With this goal in mind therefore, it is important to construct the right conceptual framework for working within this perspective. If we’re interested in state preparations then the states that are prepared result from the competing actions of the players. The players must have some reason, some preference, for the preparation of one state over another. Therefore the appropriate language is that of *preference*. The players have some preference for the states they wish to be output from a game and it is the competing preferences of the players that result in a particular output state.

Accordingly, in this paper we wish to explore the notion of preference in quantum games. It is common in game theory to associate numerical weightings, which for clarity we term payoffs, with measurement results¹ to express a preference; each player attempting to act in such a way as to maximise the numerical value of his or her payoff. Thus in classical Prisoner’s Dilemma we might envision coins initially prepared in the state (H, H) with the players each given one coin that they can flip or leave alone. The players will produce one of 4 possible output states of the coins which can then be determined (measured) and the measurement result mapped to some specific payoff. For example, if the players produce the output state (H, H) then the payoff for the players is commonly expressed as the tuple $(3, 3)$ and the output state (H, T) is associated with the payoff tuple $(0, 5)$. In classical games the act of measurement is, in general, taken to be a passive action which does not affect the output state produced by the players. The resulting matrix of payoff tuples for the possible actions of the players then determines the particular actions that will be chosen, assuming rational players.

In quantum mechanics, however, the act of measurement is anything but a passive process. If we consider an ideal von Neumann measurement then, unless the output states produced by the players are eigenstates of the measurement

¹Here we make explicit that even in a classical game the output state produced by the players in a playable game has to be measured. Of course in most classical games this can be safely ignored since measurement does not disturb the output (pre-measurement) state produced by the players.

operator, the measurement will project the output state of the players onto a new state according to some probability distribution. The quantum measurement thus furnishes an *expected* payoff for the players. As we have shown [4], if the players act in such a way as to maximize their expected payoffs, this can lead to a transformation of a game in the sense that although the players initially have some preference over the measurement eigenstates (expressed by some numerical weighting) these lead to an *induced* preference over the output states as given by the ordering of the expected payoffs. The initial weightings associated with the *measurement* results may be consistent with a game of Prisoner's Dilemma, say, but the players make their choice according to an analysis of the *expected payoffs* and may end up playing a game with the preferences of Chicken rather than Prisoner's Dilemma.

This feature of game transformation, as described above, is not unique to quantum mechanics and can occur in strictly classical games in which there is a distribution over measurement results. Such a distribution can occur, for example, if the choices of the players are communicated over a noisy channel. The important feature is that the initial preferences over the measurement results, expressed by numerical weightings, *induce* a preference on the pre-measurement output states. It is these *induced* preferences that determine the game the players *actually* play and not the initial preferences on the measurement eigenstates.

In this paper we explore a description of quantum games in terms of preferences over the pre-measurement output states. We show that a game in which there are strictly competitive preferences defined on the measurement eigenstates is equivalent to a game in which the preferences over the pre-measurement output states are defined according to a distance from some most-preferred state. As we have previously shown, preferences on output states based on distance have a nice geometric interpretation [3]. Our result here shows that any quantum game with a strictly-competitive weighting on the measurement eigenstates can be interpreted using this geometrical treatment on the pre-measurement output states.

2 General Formulation of a Discrete Quantum Game

We define a Discrete Quantum Game (DQG) as one in which the players have a finite set of actions from which to choose; the actions being unitary transformations of the input state. There will, therefore, be a finite set of possible output states produced by the players. Of course, in many quantum game formulations the players can access a continuum of possibilities (such as a general rotation of a qubit). For convenience we restrict our analysis to the case of a DQG, but the main results carry over to the continuum case in a straightforward manner.

The basic elements are the following:

- There is some initial state $|\psi_{in}\rangle$

- The players have some finite set of available operations with which they can modify the input state. We denote the operations available to player 1 by $\{\hat{\alpha}_1, \dots, \hat{\alpha}_m\}$ and those available to player 2 by $\{\hat{\beta}_1, \dots, \hat{\beta}_n\}$
- The set of possible output states that can be produced is denoted by Ψ_{out} and there are up to a total of mn possible distinct output states that can be produced. In other words $|\Psi_{out}| \leq mn$
- $\Psi_{out} \subset \Psi$ where Ψ is the set of all possible states of the physical system on which the players operate

The final element required to be able to describe this as a game is that the players have some preference over the output states that are produced. If the set of output states is written as

$$\Psi_{out} = \{|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_{mn}\rangle\}$$

then the preferences of each player are equivalent to different orderings of this set. The players thus choose an operation from their available set in order to produce their most preferred output state *given that their opponent is doing the same*. It is this ‘push-pull’ on the possible output states that leads to the production of an equilibrium state, where such an equilibrium exists. The general formulation of a DQG is shown in figure 1

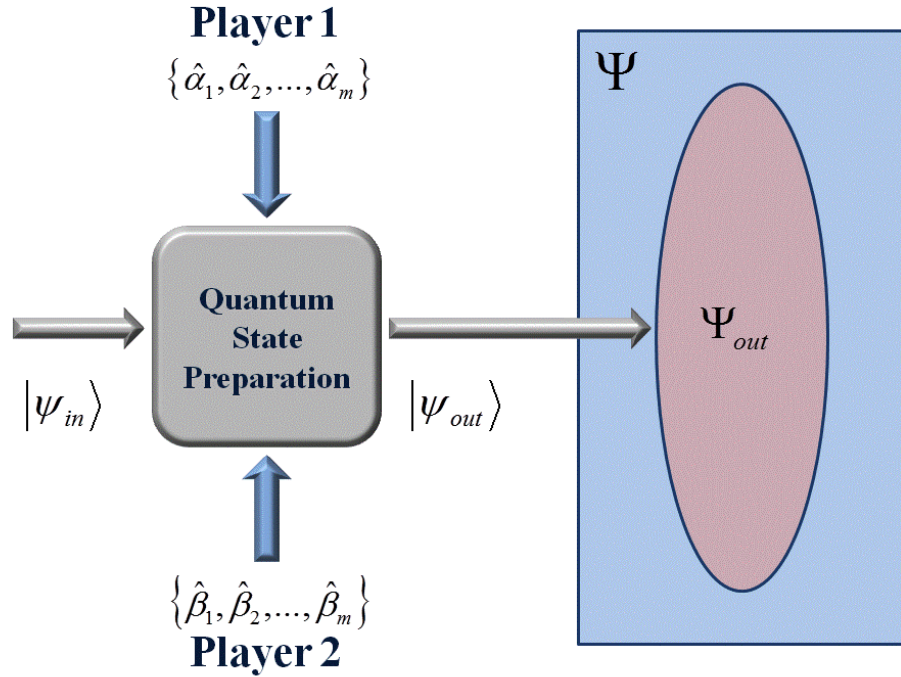


Figure 1 : *the general description of a discrete quantum game (DQG). The players each have a finite set of available unitary operations which act on some initial state. The players have competing preferences over the set of possible output states Ψ_{out} that can be prepared. The actual state $|\psi_{in}\rangle \in \Psi_{out}$ that is prepared depends on the theoretical analysis the players undertake of each others' choices based upon their knowledge of the available operations and preferences.*

2.1 Quantum Formulation of Standard Prisoner's Dilemma

As an example of the application of this formalism we consider the implementation of the standard version of Prisoner's Dilemma (PD) using 2 spin-1/2 particles. The initial state is expressed in the computational basis (taken to be eigenstates of the spin- z operator) as $|00\rangle$. The players each have a particle on which to act and the available operations are $\{\hat{\alpha}_1 = \hat{I}_1, \hat{\alpha}_2 = \hat{F}_1\}$ and $\{\hat{\beta}_1 = \hat{I}_2, \hat{\beta}_2 = \hat{F}_2\}$ for players A and B , respectively, and \hat{I} and \hat{F} are the identity and spin-flip operators acting on particles 1 and 2. There are thus 4 possible output states that can be produced which can be labelled as

$$\begin{aligned} |\psi_1\rangle &= \hat{\alpha}_1 \hat{\beta}_1 |\psi_{in}\rangle = |00\rangle \\ |\psi_2\rangle &= \hat{\alpha}_1 \hat{\beta}_2 |\psi_{in}\rangle = |01\rangle \\ |\psi_3\rangle &= \hat{\alpha}_2 \hat{\beta}_1 |\psi_{in}\rangle = |10\rangle \\ |\psi_4\rangle &= \hat{\alpha}_2 \hat{\beta}_2 |\psi_{in}\rangle = |11\rangle \end{aligned} \quad (1)$$

In PD the preferences over these output states can be expressed for players A and B as

$$\begin{aligned} A : \quad & |\psi_3\rangle \succ |\psi_1\rangle \succ |\psi_4\rangle \succ |\psi_2\rangle \\ B : \quad & |\psi_2\rangle \succ |\psi_1\rangle \succ |\psi_4\rangle \succ |\psi_3\rangle \end{aligned} \quad (2)$$

It is common to express the preferences in terms of some numerical weighting over the measurement results with respect to some measurement operator. If a measurement of spin- z is made on both particles the set of eigenstates of the measurement operator is equal to the set of possible output states that the players produce. The usual weightings for PD, given by a mapping of a numerical value to the measurement states, are

$$\begin{aligned} |00\rangle &\longrightarrow (3, 3) \\ |01\rangle &\longrightarrow (0, 5) \\ |10\rangle &\longrightarrow (5, 0) \\ |11\rangle &\longrightarrow (1, 1) \end{aligned} \quad (3)$$

There is nothing particularly quantum mechanical about this form of PD. Indeed, we could view this quantum formulation of the game as an expensive implementation of the classical version.

In this formulation of the game the possible output states that can be produced by the players are the *eigenstates* of the measurement operator. However, we are not restricted to consideration of this particular measurement. The possible output states may not be eigenstates of the subsequent measurement operator². The measurement will then yield a measurement result with a particular probability. In order to translate this to a preference on the output states we must consider some measure such as the *expected* payoffs for the players for each possible output state.

In general, therefore, a set of preferences that is *induced* on the output states via application of the mapping of numerical weights to the measurement states and consideration of the expected outcome *need not be equal* to the initial preferences over the measurement results. This is obvious from the DQG formalism since if we denote the set of measurement eigenstates as Ψ_{meas} then the cardinalities of the sets Ψ_{meas} and Ψ_{out} are only equal in special cases.

2.2 Games Using the EWL Protocol

In their ground-breaking paper on quantum games Eisert, Wilkens and Lewenstein (EWL) proposed the following structure for a game played with 2 qubits represented by a pair of spin-1/2 particles [2]. The qubits are prepared in some initial state (for convenience often taken to be the ground state $|00\rangle$ with respect to some spin direction). An entanglement operator \hat{E} is applied to the qubits and each player receives one of the particles. The players can then apply some local unitary rotation to their particle which we denote by $\hat{U}_{A(B)}$. A disentangling operator \hat{E}^{-1} is then applied to the qubits and a measurement of spin in some given direction is made on each particle. A set of weightings on the output states is given which reflect the classical game which the quantum version generalizes.

This protocol has become the *de facto* standard in treatments of quantum games (see, for example, the brief selection [6-12] which illustrate some interesting and important results obtained using this protocol). The entangling/disentangling step is to exploit the resource of entanglement whilst still allowing the players to perform local operations on their respective particle. It is a clever protocol which reduces to the classical game (described by the measurement weightings) in the appropriate limit [13]. The perception that the players only act on their own particle is, however, something of an illusion. The DQG formalism for playable games shows that in reality the players are acting on the input state, which in the case of the EWL protocol is a maximally entangled state. The EWL protocol is equivalent to a game in which the players are

²Consider, for example, a measurement of spin at some angle θ with respect to z . The set of measurement results will be the eigenstates $\{|00\rangle_\theta, |01\rangle_\theta, |10\rangle_\theta, |11\rangle_\theta\}$ the particular state obtained upon measurement being determined by a probability distribution dependent upon θ . If we establish some preference over these measurement results via a mapping to a tuple of outcomes for each eigenstate, then the expected outcome tuples for the players will induce a set of preferences on the output states. In the extreme case, $\theta = \pi/2$, each possible output state is equally preferred via this induced preference.

given the set of available operations $\{\hat{E}^{-1}\hat{U}_A\hat{E}\}$ and $\{\hat{E}^{-1}\hat{U}_B\hat{E}\}$, for players A and B , respectively. In effect in games of this structure the players act on *both* particles.

It is clear in the EWL protocol that we have only 4 measurement eigenstates but a continuum of available operations for the players, and therefore a continuum of possible output states. It is important that the available operations available to the players do not provide a map of the initial state onto the space of all possible 2 qubit states, Ψ , otherwise equilibrium cannot be attained. The operator sets $\{\hat{E}^{-1}\hat{U}_A\hat{E}\}$ and $\{\hat{E}^{-1}\hat{U}_B\hat{E}\}$ only produce a subset of the set of possible 2 qubit states. Indeed, in the case of the EWL protocol, the available operations cannot access the set of maximally entangled states of 2 qubits and so $\Psi_{out} \subset \Psi$.

The weightings on the measurement eigenstates *induce* a set of preferences on this subset via the expected payoffs for the players. It is then clear why EWL results in an enhanced equilibrium payoff for the players for Prisoner's Dilemma; we are no longer playing standard PD but a different game³. It is possible to find a classical extension of PD with correlated classical noise which gives the same enhancement of the equilibrium payoff [4].

2.3 Non-Commuting Games

The DQG treatment of the EWL protocol highlights another interesting feature of quantum games. It is possible to consider games in which the operations available to the players do not commute. In terms of games in which we have 2 qubits we then envision a game in which the players have the capability of affecting each other's particle. So we can define a non-commuting game as a game in which $[\hat{\alpha}_i, \hat{\beta}_j] \neq 0$ for at least one choice of i and j .

In the DQG formalism we can see that the players act on some initial quantum state $|\psi_{in}\rangle$ and so, in general, the players act on a single entity in quantum mechanical terms. It is only in a special class of quantum games where the Hilbert space can be split into a physically meaningful tensor product $H_A \otimes H_B$ can we envision the possibility of the players acting on 'independent' physical entities. As we have seen, in the EWL protocol the players only have the illusion of acting on separate particles; the entanglement operation giving the necessary connection so that game transformation (and enhancement of equilibrium payoff) can be achieved. The standard EWL protocol is a commuting game; the players only have access to local operations after the initial entanglement and $[\hat{E}^{-1}\hat{U}_A\hat{E}, \hat{E}^{-1}\hat{U}_B\hat{E}] = 0$ for any choice of \hat{U}_A and \hat{U}_B .

Thus, in general, in non-commuting games, the order of play matters. It is easy to construct simple game examples in which switching the order of play changes the game that is played. This is obvious from the formalism of a DQG

³It is appropriate to view quantum PD as an extension of standard PD implemented with quantum objects. If we then wish to compare quantum and classical features we must consider comparison with possible *classical extensions* of PD.

since the set of output states where B plays first is $\Psi_{out}^{AB} = \{\hat{\alpha}_i \hat{\beta}_j |\psi_{in}\rangle\}$ and the set of output states where A plays first is $\Psi_{out}^{BA} = \{\hat{\beta}_j \hat{\alpha}_i |\psi_{in}\rangle\}$ so that for non-commuting games $\Psi_{out}^{AB} \neq \Psi_{out}^{BA}$.

2.4 Time Evolution as a Quantum Game

As another illustration of the approach based on preferences and the DQG perspective consider the following simple and somewhat formal game in which the operator sets available to the players are

$$\left\{ \hat{I}, \exp \left(-it \left\{ \frac{\hat{H} + \Delta \hat{H}}{2} \right\} \right) \right\}_A$$

$$\left\{ \hat{I}, \exp \left(-it \left\{ \frac{\hat{H} - \Delta \hat{H}}{2} \right\} \right) \right\}_B \quad (4)$$

where \hat{I} is the identity, \hat{H} is a time-independent Hamiltonian and we suppose that $[\hat{H}, \Delta \hat{H}] = 0$. If our input state to the game is $|\psi(0)\rangle$ then the 4 possible output states are $\{|\psi(0)\rangle, |\psi_+(t)\rangle, |\psi_-(t)\rangle, |\psi(t)\rangle\}$ where $|\psi_{\pm}(t)\rangle = \exp \left(-it \left\{ \frac{\hat{H} \pm \Delta \hat{H}}{2} \right\} \right) |\psi(0)\rangle$ and $|\psi(t)\rangle = \exp(-i\hat{H}t) |\psi(0)\rangle$. If the players have preferences on the output states given by

$$\begin{aligned} A: & \quad |\psi_+\rangle \succ |\psi(t)\rangle \succ |\psi(0)\rangle \succ |\psi_-\rangle \\ B: & \quad |\psi_-\rangle \succ |\psi(0)\rangle \succ |\psi(t)\rangle \succ |\psi_+\rangle \end{aligned} \quad (5)$$

then the Nash equilibrium output of this game is $|\psi(t)\rangle$ which is simply the time-evolution of the initial state under the Hamiltonian \hat{H} . This is admittedly a rather contrived and artificial example, but it does illustrate the possibility of interpreting the evolution of a quantum state as the output of some 2-player game. Of course, it is clear that there will be an infinite number of possible games we can construct that yield the desired time-evolved state as the equilibrium output. If the available operations, and preferences, are suitably chosen then we may obtain an insight into quantum evolution as a ‘push-pull’ between competing physical processes. Rather speculatively, then, we might ask whether it is possible to view the path integral approach to quantum mechanics as a competition between different paths giving an interpretation of the least action principle as arising from a game-theoretic description.

If the preferences over the output states determine the game the players actually play, then in a certain sense it is these preferences that have a more fundamental character than preferences expressed as numerical weightings over some measurement results with respect to some given operator; different measurement

operators may induce different preferences on the possible output states that can be produced. By shifting the perspective to preferences over the output states, rather than preferences over some specific set of measurement results we obtain a more general description of a quantum game. The players are not restricted to obtaining their preferences on the possible output states via consideration of a specific measurement and the induced preferences over those output states, but can use whatever rule they see fit to generate their preference. The only thing that is important, as far as the analysis of the state preparation as a game is concerned, is that the players have *some* preference over the output states. *From where they get their preference over the output states is largely immaterial.*

3 Operations and Preferences

In the perspective we have outlined above a game is completely specified by

- the initial state $|\psi_{in}\rangle$
- the available operations specified by the sets $\{\hat{\alpha}_1, \dots, \hat{\alpha}_m\}$ and $\{\hat{\beta}_1, \dots, \hat{\beta}_n\}$
- an ordering, for each player, on the possible output states $|\psi_{ij}\rangle = \hat{\alpha}_i \hat{\beta}_j |\psi_{in}\rangle \in \Psi_{out}$ which expresses their preferences on this set. (Equivalently, for a given initial state, we can view the preferences as orderings on the set of operations with elements $\hat{\alpha}_i \hat{\beta}_j$)

A game can then be viewed as an algorithmic procedure which takes as inputs the initial state, the sets of operations, and the preferences over the states $|\psi_{ij}\rangle = \hat{\alpha}_i \hat{\beta}_j |\psi_{in}\rangle$ and outputs a tuple $\{\hat{\alpha}_p, \hat{\beta}_s\}$. If this algorithmic procedure does not terminate then there is no equilibrium choice for the players. It is important to note that the preferences on the *output states* alone are not sufficient to determine the game that is played. This is nicely illustrated with the following example.

Consider 2 players playing a game which has a physical implementation using 2 spin-1/2 particles. The initial state is expressed in the computational basis (taken to be eigenstates of the spin- z operator) as $|00\rangle$. The operations available to the players are $\{\hat{\alpha}_1, \hat{\alpha}_2\}$ and $\{\hat{\beta}_1, \hat{\beta}_2\}$ for players A and B , respectively. There are thus 4 possible output states that can be produced. Let us consider operations such that we have the set of output states

$$\begin{aligned}
|\psi_1\rangle &= \hat{\alpha}_1 \hat{\beta}_1 |\psi_{in}\rangle = |00\rangle \\
|\psi_2\rangle &= \hat{\alpha}_1 \hat{\beta}_2 |\psi_{in}\rangle = |01\rangle \\
|\psi_3\rangle &= \hat{\alpha}_2 \hat{\beta}_1 |\psi_{in}\rangle = |10\rangle \\
|\psi_4\rangle &= \hat{\alpha}_2 \hat{\beta}_2 |\psi_{in}\rangle = |11\rangle
\end{aligned} \tag{6}$$

If we denote the operation of flipping the spin of particle 1 by \hat{F}_1 and the same operation for particle 2 as \hat{F}_2 , then *one way* to achieve these possible output states is to give the players the choices $\{\hat{\alpha}_1 = \hat{I}, \hat{\alpha}_2 = \hat{F}_1\}$ and $\{\hat{\beta}_1 = \hat{I}, \hat{\beta}_2 = \hat{F}_2\}$ where \hat{I} is the identity operation. If the players have the preferences defined by the orderings of Ψ_{out}

$$\begin{aligned} A : & \quad |\psi_1\rangle \succ |\psi_2\rangle \succ |\psi_3\rangle \succ |\psi_4\rangle \\ B : & \quad |\psi_4\rangle \succ |\psi_3\rangle \succ |\psi_2\rangle \succ |\psi_1\rangle \end{aligned} \quad (7)$$

then it is evident that the equilibrium state $|\psi_2\rangle = \hat{\alpha}_1 \hat{\beta}_2 |\psi_{in}\rangle = |01\rangle$ is produced. However, if we retain these same preferences over the output states but consider operators such that

$$\begin{aligned} |\psi_1\rangle &= \hat{\alpha}_1 \hat{\beta}_1 |\psi_{in}\rangle = |00\rangle \\ |\psi_2\rangle &= \hat{\alpha}_2 \hat{\beta}_1 |\psi_{in}\rangle = |01\rangle \\ |\psi_3\rangle &= \hat{\alpha}_1 \hat{\beta}_2 |\psi_{in}\rangle = |10\rangle \\ |\psi_4\rangle &= \hat{\alpha}_2 \hat{\beta}_2 |\psi_{in}\rangle = |11\rangle \end{aligned} \quad (8)$$

then the equilibrium state $|\psi_3\rangle = \hat{\alpha}_1 \hat{\beta}_2 |\psi_{in}\rangle = |10\rangle$ is produced by the game. We can achieve this game by giving the players the sets of operator choices

$$\begin{aligned} A : & \quad \{\hat{\alpha}, \hat{F}_2 \hat{\alpha}\} \\ B : & \quad \{\hat{\alpha}^{-1}, \hat{\alpha}^{-1} \hat{F}_1\} \end{aligned} \quad (9)$$

where $\hat{\alpha}$ is any arbitrary unitary transformation that acts on the particles. If we think in terms of an implementation in which each player is given a ball which can be switched from a red or blue colour, with the initial state being that both balls are red, then this latter game is equivalent to giving the players the ability to play with each other's balls. It is easy to see from these particular physical instantiations of the games that they are, in fact, different games. This is perhaps more clearly seen by examining the preferences over the available operations. In the former game we have the preferences for player A over the available operations given by

$$A : \quad \hat{\alpha}_1 \hat{\beta}_1 \succ \hat{\alpha}_1 \hat{\beta}_2 \succ \hat{\alpha}_2 \hat{\beta}_1 \succ \hat{\alpha}_2 \hat{\beta}_2 \quad (10)$$

whereas in the latter game we have the preferences

$$A : \quad \hat{\alpha}_1 \hat{\beta}_1 \succ \hat{\alpha}_2 \hat{\beta}_1 \succ \hat{\alpha}_1 \hat{\beta}_2 \succ \hat{\alpha}_2 \hat{\beta}_2 \quad (11)$$

Thus whilst we may have the same preferences over the output states in both games we have different preferences over the set of available operations. It is clear from this simple example that there will be many ways to assign operations

to the players that lead to the same set of output states. The same set of preferences on output states can, therefore, represent different games. With the proviso that it is in fact a more fundamental perspective to consider preferences over the set of *operations* we shall continue to examine the preferences over the *output states* in the subsequent sections, assuming that the available set of operations has been fixed.

4 Preferences in Discrete Quantum Games

In much of the previous work on quantum game theory the preferences over the output states have largely been derived by reference to some subsequent measurement that is performed on the output states. The numerical weightings assigned to the possible measurement results then *induce* preferences on the pre-measurement output states via some measure such as the expected outcome for the players. All that is actually required for the players to determine their choice of strategy is that they have some preference on the possible pre-measurement output states that can be produced. The players (assumed rational in their choice of action) must have some foundation for the *theoretical* analysis they undertake in order to decide their choice of operation, *before any measurement or operation is performed*, and that foundation is provided by the preferences on the possible output states.

Of course each player is free to choose a preference over the output states in any arbitrary way. To illustrate this in a particularly gruesome fashion we could imagine that one player determines his preferences at random; the random numbers being generated by removing the legs from one side of a spider and noting the time it takes for the hapless spider to turn 3 circles⁴. We are not interested in such whimsical or bizarre methods for determining a preference but in those preferences that can be generated by a well-defined prescription that is related to quantities of physical interest. Examples of such prescriptions may include

- A known measurement is to be performed. Numerical weightings are assigned to the possible measurement results and the players derive their preferences by ranking the output states in order of decreasing expected outcomes. This analysis is, obviously, performed pre-measurement. This is a common way to assign preferences on the output states in a quantum game; the preferences being induced by the assignment of weights on the post-measurement states and the calculation of an expected outcome.
- A preference may be defined by ranking the output states in terms of their distance from some state $|\xi\rangle \in \Psi$. This chosen state $|\xi\rangle$ may, or may not be, a member of the set Ψ_{out} . It is necessary that the players choose to assign their preferences with respect to different states otherwise both

⁴Players are only assumed rational in their analysis of the preferences that lead to their choice of action. The players are not required to have a rational basis for their choice of preference.

players generate identical preferences over the output states and we do not have a game. Thus player A may be interested in minimizing the distance of the output states from some arbitrary state $|\xi_A\rangle$ whereas player B may minimize the distance with respect to some state $|\xi_B\rangle$

- Player A may assign his preference on the output states by requiring that the variance of the output states with respect to some measurement \hat{M}_A is minimized. Player B may seek to produce an output state that minimizes the variance with respect to some measurement \hat{M}_B . Again, it is necessary that the players seek to minimize the variance with respect to different operators otherwise they generate an identical ordering (preference) on the output states.
- The players may wish to choose an output state that maximizes the Shannon information with respect to some operator. Again the operator choices made by the players, for the purpose of defining their preference, should not be the same.

These few examples are clearly not an exhaustive list of all the interesting, and physically relevant, ways in which the players can generate their preferences on the output states. As these examples illustrate, it is necessary that the players generate a *different* ordering (preference) on the set of possible pre-measurement output states.

As a possible speculative example of this approach let us consider a quantum computation that produces some desired output state $|\lambda\rangle$. If we identify this as the Nash equilibrium state of some game between 2 players then the challenge is to find a game such that it produces the desired computation as an equilibrium output for any choice of input. The preferences over the output states may then be translated to preferences over some measurement on the output states. If we can find such a game, and such a measurement, then we have a model of a quantum computation as a 2-player game. It may be technically easier to implement this game than to build the various quantum logic gates required. Whilst it appears possible to theoretically construct such games and measurements for very simple quantum computations it seems to be highly non-trivial to demonstrate this in general.

4.1 Preferences Induced By Measurement

We now consider a situation where the players determine their preferences over the output states by analysis of the results of a subsequent measurement. As above we assume some known input state $|\psi_{in}\rangle$ upon which the players can act with a discrete set of operations specified by the sets $\{\hat{\alpha}_1, \dots, \hat{\alpha}_m\}$ and $\{\hat{\beta}_1, \dots, \hat{\beta}_n\}$. There are thus up to mn possible pre-measurement output states that can be produced given by

$$|\psi_{ij}\rangle = \hat{\alpha}_i \hat{\beta}_j |\psi_{in}\rangle$$

It is often assumed that the Hilbert space of the initial input system is the product space $H = H_A \otimes H_B$ in order to allow each player to act on a separate physical entity (the typical scenario being the space of 2 spin-1/2 particles representing 2 qubits). This division is not strictly necessary and if we allow the possibility of entanglement (either through specification of an initial state or actions allowed by the players) then, strictly speaking, we must consider the 2 particles as a single entity defined by a single state as we have described above.

- The subsequent measurement on the output states is described by the operator

$$\hat{M} = \sum_k \varphi_k |\varphi_k\rangle \langle \varphi_k| \quad (12)$$

where $\hat{M} |\varphi_k\rangle = \varphi_k |\varphi_k\rangle$ and we assume an ideal measurement. The set of measurement eigenstates will be denoted by Ψ_M .

- The measurement of \hat{M} , made on the output state $|\psi_{ij}\rangle = \hat{\alpha}_i \hat{\beta}_j |\psi_{in}\rangle$, results in the eigenstate $|\varphi_k\rangle$ with probability $|\langle \psi_{ij} | \varphi_k \rangle|^2$
- To each eigenstate $|\varphi_k\rangle$ a tuple is assigned, denoted by (ω_k^A, ω_k^B) which gives the payoff for the players, for that particular measurement result
- The expected payoffs for the players, for each possible pre-measurement output state, are given by

$$\begin{aligned} \langle \hat{O}_A \rangle_{ij} &= \sum_k \omega_k^A |\langle \psi_{ij} | \varphi_k \rangle|^2 \\ \langle \hat{O}_B \rangle_{ij} &= \sum_k \omega_k^B |\langle \psi_{ij} | \varphi_k \rangle|^2 \end{aligned} \quad (13)$$

- \hat{O}_A and \hat{O}_B are the formal Hermitian outcome operators

$$\begin{aligned} \langle \hat{O}_A \rangle &= \sum_k \omega_k^A |\varphi_k\rangle \langle \varphi_k| \\ \langle \hat{O}_B \rangle &= \sum_k \omega_k^B |\varphi_k\rangle \langle \varphi_k| \end{aligned} \quad (14)$$

- Ranking the expected outcomes in numerical order gives each an induced preference on the possible pre-measurement output states. It is these induced preferences which the players analyse in order to determine their choice of action from the sets $\{\hat{\alpha}_1, \dots, \hat{\alpha}_m\}$ and $\{\hat{\beta}_1, \dots, \hat{\beta}_n\}$ and thus the induced preferences determine the game the players *actually play*, even though the weightings assigned to the *measurement results* may reflect a different game

This structure defines induced preferences based on a calculation of the expected outcomes. One could, for example, also envision inducing preferences on the pre-measurement output states by the calculation of some expectation of a function of the output operators $\langle f(\hat{O}_A) \rangle$ and $\langle f(\hat{O}_B) \rangle$. As we have noted above, the players are free to determine their preferences in whatever way they see fit. Once their preferences have been fixed, and assuming rational players, it is then a computational matter to determine what their best choice of action is (if one exists). In games of complete information each player can thus predict the action of the other where a fixed (equilibrium) action exists.

4.2 Preferences Based on a Distance Measure

A measure that yields a geometric interpretation is that of distance. Thus we might suppose that player A wishes to get as close as possible, as determined by this distance measure, to some state $|\pi_A\rangle$ whereas player B is trying to minimize the distance of the output states to some other state $|\pi_B\rangle$. Of course, it may be that $|\pi_{A(B)}\rangle \notin \Psi_{out}$, so the players rank the possible output states in terms of increasing $|\langle \psi_{ij} | \pi_{A(B)} \rangle|^2$ if the square overlap is used as the distance measure.

Let us consider the set Ψ of all states for the physical system upon which the players act. For a DQG we clearly have $\Psi_{out} \subset \Psi$. That is, the actions available to the players do not produce a map of the input state onto the complete set of possible states of the physical system. We now consider 3 ways of defining preferences on the set of possible pre-measurement output states:

1. **P1** : We define a preference *directly* on the output states by ordering them according to their distance from some *global* most preferred state. The global maximum is the state we would most prefer to be the output given a choice over the *entire* Hilbert space of outputs Ψ . This global most preferred state will be the state that maximises the expected outcome $\langle \hat{O}_{A(B)} \rangle$ for *some* measurement and associated numerical weights.
2. **P2** : We define a preference *directly* on the output states by ordering them according to their distance from some *local* most preferred state, where this is defined as the output state from the set of possible output states that minimizes the distance to the globally most preferred state defined in preference **P1**. Thus the distance is defined strictly in terms of the possible output states which are all contained within Ψ_{out}
3. **P3** : We define a set of preferences over the measurement eigenstates of some operator, as expressed by some weightings. These then *induce* a preference over the output states by calculation of the expected outcomes for each possible output state in Ψ_{out} . Note that, if we denote the set of measurement eigenstates by Ψ_{meas} then it is possible to have $\Psi_{meas} \cap \Psi_{out} = \emptyset$

We denote the *global* state (the state in the universal set Ψ) that maximises $\langle \hat{O}_{A(B)} \rangle$ by $|\Gamma_{A(B)}\rangle$ and the *local* state (the state in the restricted set Ψ_{out}) that maximises $\langle \hat{O}_{A(B)} \rangle$ by $|\gamma_{A(B)}\rangle$ ⁵. Because the available operations of the players do not map the input state to Ψ the local maximum state does not necessarily equal the global maximum state. If the *actual* state produced by the players is $|\psi_{out}\rangle \in \Psi_{out}$ then we have

$$\begin{aligned} \langle \hat{O}_A \rangle &= \sum_k \omega_k^A |\langle \psi_{out} | \varphi_k \rangle|^2 \leq \sum_k \omega_k^A |\langle \gamma_A | \varphi_k \rangle|^2 \leq \sum_k \omega_k^A |\langle \Gamma_A | \varphi_k \rangle|^2 \\ \langle \hat{O}_B \rangle &= \sum_k \omega_k^B |\langle \psi_{out} | \varphi_k \rangle|^2 \leq \sum_k \omega_k^B |\langle \gamma_B | \varphi_k \rangle|^2 \leq \sum_k \omega_k^B |\langle \Gamma_B | \varphi_k \rangle|^2 \end{aligned} \quad (15)$$

The preference definitions **P1** and **P2** are not equivalent. This can easily be seen by consideration of the following output states $\Psi_{out} = \{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$ of a game played with a single qubit such that

$$\begin{aligned} |\psi_1\rangle &= a|0\rangle + b|1\rangle \\ |\psi_2\rangle &= b|0\rangle - a|1\rangle \\ |\psi_3\rangle &= c|0\rangle + d|1\rangle \end{aligned} \quad (16)$$

with $a, b, c, d \in \mathbb{R}$ such that $a^2 > b^2 > c^2$ and we assume the global most preferred state is $|0\rangle$. Defining a preference based on distance to this *globally* most preferred state leads to the preference ordering on the outputs

$$\mathbf{P1}: \quad |\psi_1\rangle \succ |\psi_2\rangle \succ |\psi_3\rangle \quad (17)$$

However, using the preference definition **P2** and defining a preference based on the distance to the *locally* most preferred state $|\psi_1\rangle = a|0\rangle + b|1\rangle$ yields the ordering

$$\mathbf{P2}: \quad |\psi_1\rangle \succ |\psi_3\rangle \succ |\psi_2\rangle \quad (18)$$

Let us now consider a preference measure defined by **P3** such that we have the preferences of a strictly competitive game defined on the *measurement eigenstates*. With a suitable labelling of these eigenstates we can, using numerical weightings, express the preferences of player A as

$$\begin{aligned} |\varphi_1\rangle &\longrightarrow a \\ |\varphi_j\rangle &\longrightarrow b \quad (j \neq 1) \\ a &> b \end{aligned} \quad (19)$$

⁵Note that $\hat{O}_{A(B)}$ are not specified; we merely note that such operators representing measurements exist. Indeed, one of the challenges when we work with preferences on pre-measurement states is to construct the observables and weightings that realize those preferences.

giving the ordering on the measurement eigenstates as

$$|\varphi_1\rangle \succ |\varphi_2\rangle = |\varphi_3\rangle = |\varphi_4\rangle = \dots \quad (20)$$

If we assume a ranking according to expected payoff, this preference on the (post-measurement) eigenstates induces a preference on the (pre-measurement) output states. The expected outcome for each possible output state is

$$\begin{aligned} \langle \hat{O}_A \rangle_{ij} &= \sum_k \omega_k^A |\langle \psi_{ij} | \varphi_k \rangle|^2 = a |\langle \psi_{ij} | \varphi_1 \rangle|^2 + b \sum_{k=2} |\langle \psi_{ij} | \varphi_k \rangle|^2 \\ &= (a - b) |\langle \psi_{ij} | \varphi_1 \rangle|^2 + b \end{aligned} \quad (21)$$

But this gives an ordering on the pre-measurement output states according to their the distance from the most preferred post-measurement state $|\varphi_1\rangle$. Thus the preferences on the pre-measurement states induced by assuming strictly competitive preferences on the post-measurement states, and a ranking according to the expected payoff, are the *same* preferences on the possible outputs that we obtain from **P1** when $|\varphi_1\rangle \notin \Psi_{out}$ (or **P2** when $|\varphi_1\rangle \in \Psi_{out}$). In other words, a distance ordering on the pre-measurement output states is equivalent to assuming the preferences of a strictly competitive game on the post-measurement states. We thus obtain the result that if the (globally) most preferred state of a player is an eigenstate of some Hermitian operator:

the preference defined by the distance to some state $|\xi\rangle$ is equivalent to a strictly-competitive ordering, by weighting, on the eigenstates of some Hermitian operator \hat{M} such that $|\xi\rangle$ is an eigenstate of \hat{M} that is assigned the biggest weight, where the expected outcome is the quantity the players wish to maximise

4.2.1 A simple strictly-competitive game

In order to illustrate the equivalence of distance-ordering on the outputs and a strictly-competitive ordering on the measurement results we will consider the 2-qubit example discussed above. In this case we have the set of output states and the set of measurement eigenstates given by

$$\begin{aligned} \Psi_{out} &= \{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle\} \\ \Psi_{meas} &= \{|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle, |\varphi_4\rangle\} \end{aligned} \quad (22)$$

As above we will assume that $|\varphi_1\rangle$ is the most preferred measurement state and that $|\psi_1\rangle$ will denote the most preferred output state (we can always relabel the states so that this is true). Now let us assume an ordering according to **P1**. Let us assume that we have

$$|\langle \Gamma_A | \psi_1 \rangle|^2 > |\langle \Gamma_A | \psi_2 \rangle|^2 > |\langle \Gamma_A | \psi_3 \rangle|^2 > |\langle \Gamma_A | \psi_4 \rangle|^2 \quad (23)$$

where $|\Gamma_A\rangle$ is the global maximum state, or the state $|\Gamma_A\rangle \in \Psi$ that is the most preferred state out of all possible states on the Hilbert space. This is equivalent to an ordering of the output states given by

$$|\psi_1\rangle \succ |\psi_2\rangle \succ |\psi_3\rangle \succ |\psi_4\rangle$$

Now let us consider the preferences defined by **P3**. In this case we have, as above, that

$$\begin{aligned} |\varphi_1\rangle &\longrightarrow a \\ \{|\varphi_2\rangle, |\varphi_3\rangle, |\varphi_4\rangle\} &\longrightarrow b \\ a &> b \end{aligned} \quad (24)$$

so that the preferences are, *initially*, defined on the measurement set Ψ_{meas} . The expected payoffs are given by

$$\langle \hat{O}_A \rangle_j = a |\langle \psi_j | \varphi_1 \rangle|^2 + b (1 - |\langle \psi_j | \varphi_1 \rangle|^2) \quad j = 1, 2, 3, 4 \quad (25)$$

However, in the perspective of **P3**, the global maximum state $|\Gamma_A\rangle$ is just the most preferred state from Ψ_{meas} . Thus we have that $|\Gamma_A\rangle = |\varphi_1\rangle$. Further, by assumption we have that

$$|\langle \Gamma_A | \psi_1 \rangle|^2 > |\langle \Gamma_A | \psi_2 \rangle|^2 > |\langle \Gamma_A | \psi_3 \rangle|^2 > |\langle \Gamma_A | \psi_4 \rangle|^2 \quad (26)$$

which is equivalent to the ordering

$$|\langle \varphi_1 | \psi_1 \rangle|^2 > |\langle \varphi_1 | \psi_2 \rangle|^2 > |\langle \varphi_1 | \psi_3 \rangle|^2 > |\langle \varphi_1 | \psi_4 \rangle|^2 \quad (27)$$

which gives the ordering of the *expected* payoffs

$$\langle \hat{O}_A \rangle_1 > \langle \hat{O}_A \rangle_2 > \langle \hat{O}_A \rangle_3 > \langle \hat{O}_A \rangle_4 \quad (28)$$

which is, of course, equivalent to an ordering on the output states of

$$|\psi_1\rangle \succ |\psi_2\rangle \succ |\psi_3\rangle \succ |\psi_4\rangle \quad (29)$$

Thus, for this strictly-competitive example, the preference definitions **P1** and **P3** are entirely equivalent; in both cases we obtain an ordering *of the output states* that is dependent of the distance of the output states from the most preferred global state, where distance is expressed by the projection $|\langle \psi_j | \varphi_1 \rangle|^2$. Note that in **P1** this preference is defined directly on the output states whereas in **P3** this preference is *induced* on the output states. Thus, for this example, defining the preferences by a preference over the *measurement* results as

$$|\varphi_1\rangle \succ |\varphi_2\rangle = |\varphi_3\rangle = |\varphi_4\rangle \quad (30)$$

yields an ordering of the *output states* that is the same ordering obtained from **P1**.

4.2.2 Preferences Based on Variances

Here we suppose that player A assigns his preference on the (pre-measurement) output states by requiring that the variance of these output states with respect to some subsequent measurement \hat{M}_A is minimized. Player B seeks to produce an output state that minimizes the variance with respect to some measurement \hat{M}_B . It is important to emphasize that the players must have some foundation upon which to base their theoretical analysis leading to their choice of ‘move’ in the game. Clearly this theoretical analysis is undertaken *before* any measurement is actually performed. Equally clearly, any physical instantiation of the game (that is, to actually play the game with physical objects and processes) will require *some* measurement to be performed. Here we are supposing that the foundation is supplied by a preference for minimizing variances of the pre-measurement output states with respect to specific observables.

For convenience we shall suppose that \hat{M}_A and \hat{M}_B are complementary observables. If the eigenstates of these operators are $|\varphi_i^A\rangle$ and $|\varphi_j^B\rangle$, respectively, then the eigenstates are related by

$$|\varphi_i^A\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^N \exp(i\theta_j) |\varphi_j^B\rangle \quad (31)$$

Player A is thus wishing to produce an output state in a game that has the smallest distance to an eigenstate of \hat{M}_A . Equivalently player A wishes to maximise the distance to eigenstates of \hat{M}_B . The players thus generate opposite preferences based on distance to some preferred set of states. We can realize these preferences by setting up a strictly-competitive game as we have discussed above.

5 Discussion

In this paper we have considered a new perspective on quantum games by considering preferences on pre-measurement output states produced by the players. Once the operations (‘moves’) available to the players are fixed then, for a given input state, these pre-measurement preferences determine the game that is played. In typical formulations of quantum games where weightings are assigned to measurement results we have demonstrated that these *induce* preferences on the pre-measurement output states [4]. It is these *induced* preferences the players use to determine their choice of strategy. In this sense, then, consideration of pre-measurement preferences is more fundamental since these determine the actual game that the players play.

We have described a formalism for a discrete quantum game defined by the pre-measurement preferences (for a given input state and set of available operations). Knowledge of these is sufficient to determine the choice of strategy and thus specifies the game. For a 2-player DQG in which there are mn possible pre-measurement output states the number of possible games that can be played for a given input state and set of available operations is $\sim (mn!)^2$ since a player’s

preference is simply a particular ordering of the set Ψ_{out} . There are choices of ordering (preference) which are more physically appealing than others and we have discussed examples of these. If we wish to actually play a game then some measurement must be performed. In general then, the challenge is to find a measurement and weightings of the measurement results which will induce the pre-measurement preferences. We have shown that for 2 important and physically relevant choices of preferences on the pre-measurement output states, a minimization of distance and variance, the playable game can be realized by a strictly competitive weighting on the measurement results.

If the players wish to minimize the distance to the states $|\pi_A\rangle$ and $|\pi_B\rangle$, respectively, then this is equivalent to player A wishing to maximise *distinguishability* of the the output state of the game from the state $|\pi_B\rangle$ and vice versa for player B . Playing such a game would then, in an appropriate sense, produce an equilibrium state (where one exists) that optimises the distinguishability from both $|\pi_A\rangle$ and $|\pi_B\rangle$ for the given set of output states that is produced.

In our previous work we have considered a distance based measure on output states and shown its importance in strictly competitive games. Here we have shown that a distance based preference measure on pre-measurement output states is *equivalent* to a strictly-competitive game defined on the measurement results. If a particular quantum computation can be identified as the equilibrium state of a game with a distance measure then we have a simple methodology for physically constructing such a computation as a strictly competitive game on measurement results.

6 References

1. D. Meyer, Quantum Strategies, *Physical Review Letters*, **82**, pages 1052-1055.(1999)
2. J. Eisert, M. Wilkens, and M. Lewenstein, Quantum Games and Quantum Strategies, *Physical Review Letters*, **83**, pages 3077-3080 (1999).
3. F.S. Khan and S.J.D .Phoenix, Gaming the Quantum, *Journal of Quantum Information and Computing*, **13(3-4)**, 231-244 (2013)
4. S.J.D .Phoenix and F.S. Khan, The Role of Correlation in Quantum and Classical Games, *Fluctuation and Noise Letters*, **12**, 1350011 (2013)
5. F.S. Khan and S.J.D .Phoenix, Mini-maximizing Two Qubit Quantum Computations, *Journal of Quantum Information and Computing*, **12(12)**, 3807-3819 (2013)
6. R. Cleve, P. Hoyer, B. Toner and J. Watrous, Consequences and Limits of Nonlocal Strategies, preprint available at <http://arxiv.org/abs/quant-ph/0404076v2> (2004)

7. J. Shimamura, S. K. Özdemir, F. Morikoshi and N. Imoto, Entangled states that cannot reproduce original classical games in their quantum version, *Physics Letters A*, **328**, 20-25 (2004)
8. A. Iqbal and S. Weigert, Quantum correlation games, *Journal of Physics A: Math. & Gen.* **37**, 5873-5885 (2004).
9. A.P. Flitney, D. Abbott D, Quantum games with decoherence, *Journal of Physics A: Mathematical and General* **38**: 449 (2005)
10. C. F. Lee and N. Johnson, Efficiency and Formalism of Quantum Games, *Physical. Review A*, **67**, 022311 (2003)
11. T. Cheon and I. Tsutsui, Classical and Quantum Contents of Solvable Game Theory on Hilbert Space, *Physics Letters A* **348** 147-152 (2006)
12. S. E. Landsburg, Nash Equilibria in Quantum Games, *Proceedings of the American Mathematical Society*, **139**, 4423-4434, (2011)
13. S. A. Bleiler, A Formalism for Quantum Games I - Quantizing Mixtures, preprint at <http://arxiv.org/abs/0808.1389>, 2008.